

EIGENVALUE RATIOS OF NONNEGATIVELY CURVED GRAPHS

SHIPING LIU AND NORBERT PEYERIMHOFF

ABSTRACT. We prove an optimal eigenvalue ratio estimate for finite weighted graphs satisfying the curvature-dimension inequality $CD(0, \infty)$. This estimate is independent of the size of the graph and provides a general method to obtain higher-order spectral estimates. The operation of taking Cartesian products is shown to be an efficient way for constructing new weighted graphs satisfying $CD(0, \infty)$. We also discuss a multi-way isoperimetric constant ratio estimate and related topics about expanders.

1. INTRODUCTION

Exploring the influence of eigenvalues on graph structures is one of the central topics in spectral graph theory, see e.g. [1], [7], [8], [9], [26]. In this area, the first nonzero (normalized or non-normalized) Laplacian eigenvalue and the Cheeger constant play a fundamentally important role and their close relations have found tremendous applications in both theoretical and applied fields. Recently, the understanding of the higher-order versions of them, i.e. the k -th nonzero Laplacian eigenvalues and the k -way isoperimetric constants have been improved significantly due to the work of Miclo [24], Lee-Gahran-Trevisan [20], Kwok et al. [18], and also see, e.g., [27], [22], [25].

In particular, Mimura [25] proved an upper bound estimate for the ratio of (consecutive) multi-way isoperimetric constants on vertex-transitive graphs. As commented there, similar results for eigenvalue ratios seems to be open.

In this paper, we establish an eigenvalue ratio estimate for an interesting class of nonnegatively curved graphs. More precisely, we consider the class of graphs satisfying the curvature-dimension inequality $CD(0, \infty)$ in the sense of Bakry-Émery [3], firstly studied for graphs by Lin-Yau [21]. This estimate is achieved by an approach that has been carried out for Riemannian manifolds in [23]. Its optimality in several aspects will be confirmed via various examples.

We now introduce the necessary technical terms and describe the results more precisely. We are working on a general structure denoted by the pair (G, μ) . $G = (V, E, w)$ stands for an undirect weighted finite connected graph, where V and E are the sets of vertices and edges, respectively, and w is a function describing the edge weights, i.e., to two vertices $x, y \in V$ with $e = (x, y) \in E$ we associate a positive symmetric weight w_{xy} . Alternatively, we write $x \sim y$ for an edge $e = (x, y)$ in E . We say the graph is unweighted,

if $w_{xy} = 1$ for any $e = (x, y) \in E$. Let $d_x := \sum_{y, y \sim x} w_{xy}$ be the degree of a vertex x and $d_G := \max_{x \in V} d_x$ be the maximal degree of the graph G . $\mu : V \rightarrow \mathbb{R}_{>0}$ stands for a finite positive measure assigned to the vertex set V of the graph G .

For any function $f : V \rightarrow \mathbb{R}$ and any vertex $x \in V$, the associated Laplacian Δ is defined as

$$\Delta f(x) := \frac{1}{\mu(x)} \sum_{y, y \sim x} w_{xy} (f(y) - f(x)).$$

This operator is called μ -Laplacian in [4]. The normalized and the non-normalized Laplacian are contained in this general setting as the following special cases:

- non-normalized Laplacian: if $\mu(x) = 1 \forall x \in V$ ($\mu = \mathbf{1}_V$ for short);
- normalized Laplacian: if $\mu(x) = d_x \forall x \in V$ ($\mu = \mathbf{d}_V$ for short).

Then the curvature-dimension inequality can be formulated purely via this operator (Definition 2.2).

Given two weighted graphs G_1 and G_2 , we denote their Cartesian product by $G_1 \times G_2$. We show that the class of graphs satisfying $CD(0, \infty)$ is actually rich since, in many interesting cases, new examples can be constructed via taking Cartesian products. Here is the general result (which is stated later again as Theorem 2.5).

Theorem 1.1. *If $(G_1, \mathbf{1}_{V_1})$ and $(G_2, \mathbf{1}_{V_2})$ satisfy $CD(K_1, n_1)$ and $CD(K_2, n_2)$ respectively, then $(G_1 \times G_2, \mathbf{1}_{V_1 \times V_2})$ satisfies $CD(K_1 \wedge K_2, n_1 + n_2)$.*

Here we used the notion $K_1 \wedge K_2 := \min\{K_1, K_2\}$. The above estimate is optimal at least for the Cartesian product of a graph G with itself (Remark 2.9). Theorem 1.1 can be extended to include the case of regular graphs with normalized Laplacian operators (Remark 2.8). In particular, the property of satisfying $CD(0, \infty)$ is preserved when taking Cartesian product in many cases.

We call λ an eigenvalue of Δ if there exists a function $f \not\equiv 0$ such that $\Delta f = -\lambda f$. We order the eigenvalues of Δ with multiplicity as follows

$$0 = \lambda_1(G, \mu) < \lambda_2(G, \mu) \leq \dots \leq \lambda_N(G, \mu) \leq 2D_G^{non}, \quad (1)$$

where N is the number of vertices in V . The following two quantities D_G^{non} and D_G^{nor} appear naturally in our arguments.

$$D_G^{non} := \max_{x \in V} \frac{\sum_{y, y \sim x} w_{xy}}{\mu(x)}, \quad \text{and} \quad D_G^{nor} := \max_{x \in V} \max_{y, y \sim x} \frac{\mu(x)}{w_{xy}}.$$

Observe that on an unweighted graph, in either of the cases $\mu = \mathbf{1}_V$ or $\mu = \mathbf{d}_V$ we always have $D_G^{non} D_G^{nor} = d_G$.

We prove the following eigenvalue ratio estimate (stated below again as Theorem 3.4). Note that the result does not depend on the size of graphs.

Theorem 1.2. *For any graph (G, μ) satisfying $CD(0, \infty)$ and any natural number $k \geq 2$, we have*

$$\lambda_k(G, \mu) \leq \left(\frac{20\sqrt{2}e}{e-1} \right)^2 D_G^{non} D_G^{nor} k^2 \lambda_2(G, \mu). \quad (2)$$

We will present examples which show that in the above estimate the order of k is optimal and the dependence on $D_G^{non} D_G^{nor}$ is necessary and optimal. We also find a sequence of graphs with size tending to infinity, each of which has only two vertices violating $CD(0, \infty)$ and which does not have the property (2).

Theorem 1.2 can be used as a general source to derive various interesting higher-order estimates between geometric invariants and spectra. We use it to show a higher-order Buser inequality (Theorem 3.10), which states a strengthened relation between $\lambda_k(G, \mu)$ and the k -way isoperimetric constant $h_k(G, \mu)$ (Definition 3.1). By extending the inequalities of Alon-Milman [1], we prove a higher-order eigenvalue-diameter estimate (Theorem 3.14), which compares nicely with the celebrated Cheng estimate (Corollary 2.2 in [6]) for compact Riemannian manifolds with nonnegative Ricci curvature. Both two inequalities seem to be the first ones of its kind established in the graph setting.

In combination with the higher-order Cheeger inequalities of Lee-Oveis Gharan-Trevisan [20], the eigenvalue ratio estimate implies naturally the following the multi-way isoperimetric constant ratio estimate (stated below again as Theorem 4.1).

Theorem 1.3. *There exists a universal constant C such that for any graph (G, μ) satisfying $CD(0, \infty)$ and any natural number $k \geq 2$ we have*

$$h_k(G, \mu) \leq C D_G^{non} D_G^{nor} k \sqrt{\log k} h_2(G, \mu). \quad (3)$$

We use an example by Mimura [25] to show that the dependence of $D_G^{non} D_G^{nor}$ in the above estimate is necessary and optimal. For graphs with bounded genus and satisfying $CD(0, \infty)$, we obtain an improvement of Theorem 1.3 with optimal order of k (Corollary 4.2). As in [25], this ratio estimate is closely related with the topics about (multi-way) expanders.

We organize the rest of the article as follows. In Section 2, we discuss in detail the curvature-dimension inequality and introduce several interesting examples for later use. In Section 3, we show the eigenvalue ratio estimate and present applications. In Section 4, we discuss the multi-way isoperimetric constant ratio estimate and related topics about expanders. Finally in the Appendix, we give more details about the curvature dimension inequality calculations in some examples and also a self-contained proof of Buser's inequality for graphs satisfying $CD(0, \infty)$.

2. CURVATURE-DIMENSION INEQUALITY

The curvature-dimension inequality (CD-inequality for short) was introduced by Bakry-Émery [3] as a substitute of the lower Ricci curvature bound of the underlying space. It was first studied on graphs by Lin-Yau [21], see also [16], [10]. Recall that the operators Γ and Γ_2 are defined iteratively as follows.

Definition 2.1. For any two functions $f, g : V \rightarrow \mathbb{R}$, we define

$$\Gamma(f, g) := \frac{1}{2} \{ \Delta(fg) - f\Delta g - g\Delta f \}, \quad (4)$$

and

$$\Gamma_2(f, g) := \frac{1}{2} \{ \Delta\Gamma(f, g) - \Gamma(f, \Delta g) - \Gamma(g, \Delta f) \}. \quad (5)$$

We also write $\Gamma(f) := \Gamma(f, f)$ and $\Gamma_2(f) := \Gamma_2(f, f)$ for short. In particular, by the definition above we have for any $x \in V$ and any f, g

$$\Gamma(f, g)(x) = \frac{1}{2\mu(x)} \sum_{y, y \sim x} w_{xy} (f(y) - f(x))(g(y) - g(x)). \quad (6)$$

A useful fact is the following summation by part formula,

$$\sum_{x \in V} \mu(x) \Gamma(f, g)(x) = - \sum_{x \in V} \mu(x) f(x) \Delta g(x), \quad (7)$$

and also

$$\Gamma(f, g) \leq \sqrt{\Gamma(f)} \sqrt{\Gamma(g)}. \quad (8)$$

Rewriting (4) provides the following chain rule,

$$\Delta(f^2) = 2\Gamma(f) + 2f\Delta(f). \quad (9)$$

Definition 2.2. Let $K \in \mathbb{R}$ and $n \in \mathbb{N} \cup \{0\}$. We say that (G, μ) satisfies the CD-inequality $CD(K, n)$ if for any functions f and any vertex x , the following inequality holds,

$$\Gamma_2(f)(x) \geq \frac{1}{n} (\Delta f(x))^2 + K\Gamma(f)(x). \quad (10)$$

In particular, we say that (G, μ) satisfies $CD(0, \infty)$ if for any functions f we have $\Gamma_2(f) \geq 0$.

2.1. Examples of graphs satisfying $CD(0, \infty)$. We will be mainly concerned with the class of graphs satisfying $CD(0, \infty)$. From (5), we see Γ_2 is a symmetric bilinear form. At every vertex $x \in V$, we denote the matrix corresponding to $\Gamma_2(f, g)(x)$ by $\Gamma_2(x)$. Let $B_2(x) := \{y \in V : \text{dist}(y, x) \leq 2\}$, where dist stands for the usual shortest-path metric on V . Then $\Gamma_2(x)$ is a symmetric matrix of size $|B_2(x)| \times |B_2(x)|$. A graph (G, μ) satisfies $CD(0, \infty)$ if and only if $\Gamma_2(x)$ is positive-semidefinite at every vertex $x \in V$. Observe that $\Gamma_2(x) + \text{Id}$ is a stochastic matrix since $\Gamma_2(x)\mathbf{c} = 0$ for any constant vector \mathbf{c} . In particular, if all the diagonal entries are nonnegative

and all the off-diagonal entries are nonpositive, then the matrix $\Gamma_2(x)$ is diagonally dominant and hence positive-semidefinite.

Next we present some usefulexamples.

Example 2.3. Consider the triangle graph Δ_{xyz} with positive edge weights a, b, c , as shown in Figure 1. Assign a measure μ to the vertices such that $\mu(x) := C, \mu(y) := B, \mu(z) := A$.

- Normalized case: Suppose $A = b + c, B = a + c, C = a + b$. Then (Δ_{xyz}, μ) satisfies $CD(0, \infty)$.
- Non-normalized case: Suppose $A = B = C = 1$. Then (Δ_{xyz}, μ) does not always satisfy $CD(0, \infty)$. If in particular $a = c$, it satisfies $CD(0, \infty)$. But when $a = 1, c = 1/b$, it does not satisfy $CD(0, \infty)$ if b is large/small enough. In fact, when $b \geq 5.01$ or $b \leq 0.12$, the symmetric curvature matrix $\Gamma_2(x)$ has a negative eigenvalue.

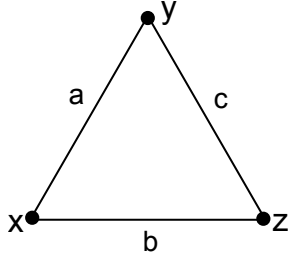


FIGURE 1. Triangle

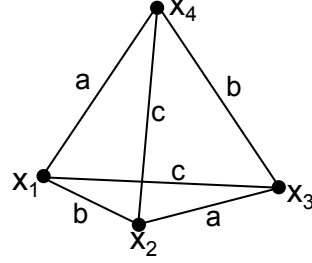


FIGURE 2. Tetrahedron

Example 2.4. Consider the tetrahedron graph T_4 with positive edge weights a, b, c as shown in Figure 2. Observe that this graph is regular, i.e., $d_{x_i} = a + b + c$ is a constant for every i . Assign a measure μ on the vertices such that $\mu(x_i) = A$ for all i , where A is a positive constant. (Note that this includes both the cases of normalized and non-normalized Laplacians.) Then (T_4, μ) always satisfies $CD(0, \infty)$.

The details about the curvature matrix Γ_2 of the triangle graph and the tetrahedron graph are given in Appendix A.1. For the normalized case, the curvature of unweighted triangle graphs was calculated in Proposition 1.6 of [21], and the curvature of general unweighted complete graphs was calculated in Proposition 3 of [16].

In fact, the tetrahedron graph in Figure 2 belongs to a large class of graphs called Ricci flat graphs with consistent edge weights. The concept of a Ricci flat graph was introduced by Chung-Yau [11] and that of consistent edge weights was further introduced in Bauer et al. [4]. We refer the reader to [11, 4] for the precise definitions. Every graph in this class is a regular graph (in fact both its unweighted and weighted degree are constant) and satisfies $CD(0, \infty)$ if we assign a measure μ such that $\mu(x) = A$ for all vertices x (see [21, 11] for the unweighted case, the weighted case follows from the same

calculations). In particular, every abelian Cayley graph is Ricci flat and hence satisfies $CD(0, \infty)$.

2.2. CD-inequalities of Cartesian product graphs. In this subsection we discuss a method for constructing new graphs satisfying certain CD-inequalities from known examples, that is, taking the Cartesian product.

Given two (possibly infinite) graphs $G_1 = (V_1, E_1, w)$ and $G_2 = (V_2, E_2, \bar{w})$, their Cartesian product $G_1 \times G_2 = (V_1 \times V_2, E_{12}, w^{12})$ is a weighted graph with vertex set $V_1 \times V_2$ and edge set E_{12} given by the following rule. Two vertices $(x_1, y_1), (x_2, y_2) \in V_1 \times V_2$ are connected by an edge in E_{12} if

$$x_1 = x_2, y_1 \sim y_2 \text{ in } E_2 \quad \text{or} \quad x_1 \sim x_2 \text{ in } E_1, y_1 = y_2.$$

In the first case above we chose the edge weight to be $\bar{w}_{y_1 y_2}$ and in the second case $w_{x_1 x_2}$.

We have the following result.

Theorem 2.5. *If $(G_1, \mathbf{1}_{V_1})$ and $(G_2, \mathbf{1}_{V_2})$ satisfy $CD(K_1, n_1)$ and $CD(K_2, n_2)$, respectively, then $(G_1 \times G_2, \mathbf{1}_{V_1 \times V_2})$ satisfies $CD(K_1 \wedge K_2, n_1 + n_2)$.*

Let $f : V_1 \times V_2 \rightarrow \mathbb{R}$ be a function on the product graph. For fixed $y \in V_2$, we will write $f_y(\cdot) := f(\cdot, y)$ as a function on V_1 . Similarly, $f^x(\cdot) := f(x, \cdot)$. The following lemma is crucial for the proof of Theorem 2.5.

Lemma 2.6. *For any function $f : V_1 \times V_2 \rightarrow \mathbb{R}$ and any $(x, y) \in V_1 \times V_2$, we have*

$$\Gamma_2(f)(x, y) \geq \Gamma_2(f_y)(x) + \Gamma_2(f^x)(y), \quad (11)$$

where the operators Γ_2 are understood to be on different graphs according to the functions they are acting on.

Proof. For simplicity, we will denote by x_i a neighbor of $x \in V_1$, and write shortly $w_i := w_{xx_i}$ in this proof. Similar notions are used for $y \in V_2$ and \bar{w} .

Recall $2\Gamma_2(f)(x, y) = \Delta\Gamma(f)(x, y) - 2\Gamma(f, \Delta f)(x, y)$. By definition, we have

$$\begin{aligned} \Delta\Gamma(f)(x, y) &= \sum_{x_i \sim x} w_i (\Gamma(f)(x_i, y) - \Gamma(f)(x, y)) \\ &\quad + \sum_{y_k \sim y} \bar{w}_k (\Gamma(f)(x, y_k) - \Gamma(f)(x, y)) := L_1 + L_2. \end{aligned}$$

For the first term L_1 , we calculate

$$\begin{aligned} L_1 &= \sum_{x_i \sim x} w_i [\Gamma(f_y)(x_i) + \Gamma(f^x)(y) - \Gamma(f_y)(x) - \Gamma(f^x)(y)] \\ &= \Delta\Gamma(f_y)(x) + \frac{1}{2} \sum_{x_i \sim x} \sum_{y_k \sim y} w_i \bar{w}_k [(f(x_i, y_k) - f(x_i, y))^2 \\ &\quad - (f(x, y_k) - f(x, y))^2]. \end{aligned}$$

Similarly, we obtain

$$L_2 = \Delta\Gamma(f^x)(y) + \frac{1}{2} \sum_{y_k \sim y} \sum_{x_i \sim x} \bar{w}_k w_i [(f(x_i, y_k) - f(x, y_k))^2 - (f(x_i, y) - f(x, y))^2].$$

Furthermore, we have

$$\begin{aligned} 2\Gamma(f, \Delta f)(x, y) &= \sum_{x_i \sim x} w_i (f(x_i, y) - f(x, y)) (\Delta f(x_i, y) - \Delta f(x, y)) + \\ &\quad \sum_{y_k \sim y} \bar{w}_k (f(x, y_k) - f(x, y)) (\Delta f(x, y_k) - \Delta f(x, y)) \\ &:= G_1 + G_2. \end{aligned}$$

Then for the term G_1 , we have

$$\begin{aligned} G_1 &= \sum_{x_i \sim x} w_i (f(x_i, y) - f(x, y)) (\Delta f_y(x_i) + \Delta f^{x_i}(y) - \Delta f_y(x) - \Delta f^x(y)) \\ &= 2\Gamma(f_y, \Delta f_y)(x) + \sum_{x_i \sim x} \sum_{y_k \sim y} w_i \bar{w}_k (f(x_i, y) - f(x, y)) \times \\ &\quad (f(x_i, y_k) - f(x_i, y) - f(x, y_k) + f(x, y)). \end{aligned}$$

Similarly, we also have

$$\begin{aligned} G_2 &= 2\Gamma(f^x, \Delta f^x)(y) + \sum_{y_k \sim y} \sum_{x_i \sim x} \bar{w}_k w_i (f(x, y_k) - f(x, y)) \times \\ &\quad (f(x_i, y_k) - f(x, y_k) - f(x_i, y) + f(x, y)). \end{aligned}$$

Observing the fact that

$$\begin{aligned} &(f(x_i, y_k) - f(x, y_k))^2 - (f(x_i, y) - f(x, y))^2 \\ &= (f(x_i, y_k) - f(x, y_k) - f(x_i, y) + f(x, y))^2 + \\ &\quad 2(f(x_i, y_k) - f(x, y_k) - f(x_i, y) + f(x, y))(f(x_i, y) - f(x, y)), \end{aligned}$$

We arrive at

$$L_2 - \Delta\Gamma(f^x)(y) - (G_1 - 2\Gamma(f_y, \Delta f_y)(x)) \geq 0, \quad (12)$$

and

$$L_1 - \Delta\Gamma(f_y)(x) - (G_2 - 2\Gamma(f^x, \Delta f^x)(y)) \geq 0. \quad (13)$$

This completes the proof. \square

Remark 2.7. The intuition of the above calculation is that the mixed terms are "flat". In fact, Lemma 2.6 still holds if we replace $\Gamma_2(f)$ by $\tilde{\Gamma}_2(f) := \frac{1}{2}\Delta\Gamma(f) - \Gamma\left(f, \frac{\Delta(f^2)}{2f}\right)$. Explicitly, for any positive function $f : V_1 \times V_2 \rightarrow \mathbb{R}$ and any $(x, y) \in V_1 \times V_2$, we have

$$\tilde{\Gamma}_2(f)(x, y) \geq \tilde{\Gamma}_2(f_y)(x) + \tilde{\Gamma}_2(f^x)(y). \quad (14)$$

The proof is done in a similar way. The operator $\tilde{\Gamma}_2$ was introduced in Bauer et al. [4] to define a modification of the CD-inequality, called exponential curvature-dimension inequality $CDE(K, n)$ (see Definition 3.9 in [4]). Under the assumption of their new notion of curvature lower bound, they prove Li-Yau type gradient estimates (dimension-dependent) for the heat kernels on graphs.

Proof of Theorem 2.5. By Lemma 2.6, we have for any function $f : V_1 \times V_2 \rightarrow \mathbb{R}$ and any $(x, y) \in V_1 \times V_2$,

$$\begin{aligned} \Gamma_2(f)(x, y) &\geq \Gamma_2(f_y)(x) + \Gamma_2(f^x)(y) \\ &\geq \frac{1}{n_1}(\Delta f_y(x))^2 + \frac{1}{n_2}(\Delta f^x(y))^2 + K_1\Gamma(f_y)(x) + K_2\Gamma(f^x)(y) \\ &\geq \frac{1}{n_1 + n_2}(\Delta f_y(x) + \Delta f^x(y))^2 + K_1 \wedge K_2(\Gamma(f_y)(x) + \Gamma(f^x)(y)). \end{aligned} \quad (15)$$

In the last inequality above we used Young's inequality. Recalling the facts $\Delta f_y(x) + \Delta f^x(y) = \Delta f(x, y)$ and $\Gamma(f_y)(x) + \Gamma(f^x)(y) = \Gamma(f)(x, y)$, we complete the proof. \square

Remark 2.8. We can have more flexibility concerning the measures assigned to vertices. Suppose the vertex measures assigned to G_1, G_2 and $G_1 \times G_2$ take the constant values μ_1, μ_2 and μ_{12} on each vertex, respectively, then the modified conclusion of Theorem 2.5 is that $(G_1 \times G_2, \mu_{12})$ satisfies

$$CD\left(\frac{1}{\mu_{12}^2}(\mu_1^2 K_1 \wedge \mu_2^2 K_2), \frac{\mu_{12}^2}{\mu_1^2 \mu_2^2}(\mu_2^2 n_1 + \mu_1^2 n_2)\right). \quad (16)$$

This modification covers the case of normalized Laplacians on regular graphs. In particular, if both (G_1, μ_1) and (G_2, μ_2) satisfy $CD(0, \infty)$, then $(G_1 \times G_2, \mu_{12})$ also satisfies $CD(0, \infty)$.

Remark 2.9. The estimates of the CD-inequality in Theorem 2.5 (in fact also (16)) are tight at least for the Cartesian product of a graph G with itself. That is, if G satisfies $CD(K, n)$ precisely (i.e., the given n, K in combination are largest possible), then the CD-inequality in Theorem 2.5 (or in (16)) is optimal for $G \times G$. This can be seen as follows. First note that this tightness depends on that of (12), (13) and (15). By assumption, there exists a function f on the graph G and a vertex x of the graph such that

$$\Gamma_2(f)(x) = \frac{1}{n}(\Delta f(x))^2 + K\Gamma(f)(x),$$

with $\Gamma(f)(x) \neq 0$. We can then choose a particular function F on $G \times G$ such that (i) $F(x, x) = f(x)$; (ii) $F(x_i, x) = f(x_i)$ for all neighbors x_i of x in G ; (iii) $F(x, x_k) = f(x_k)$ for all neighbors x_k of x in G ; (iv) $F(x_i, x_k) = F(x, x_k) + F(x_i, x) - F(x, x)$. For such a F the equalities in (12) and (13) are attained at (x, x) and $\Delta F_x(x) = \Delta F^x(x)$, hence the equality in (15) is

also attained. Therefore we obtain

$$\Gamma_2(F)(x, x) = \frac{1}{2n}(\Delta F(x, x))^2 + K\Gamma(F)(x, x),$$

which confirms the postulated tightness.

2.3. Bakry-Émery type gradient estimate. Consider the following continuous time heat equation,

$$\begin{cases} \frac{\partial}{\partial t}u(x, t) = \Delta u(x, t), \\ u(x, 0) = f(x). \end{cases} \quad (17)$$

This is in fact a matrix differential equation. Its solution $u : V \times [0, \infty) \rightarrow \mathbb{R}$ can be written as $u(x, t) = P_t f(x)$ where $P_t := e^{t\Delta}$. Let us choose an orthonormal basis $\{\psi_i\}_{i=1}^N$ of the space $l^2(V)$ (the function space defined by the inner product $(f, g)_\mu := \sum_{x \in V} f(x)g(x)\mu(x)$), consisting of eigenfunctions of Δ . One can derive the following properties from the definition.

Proposition 2.10 (see, e.g., [4, 8]). *The operator $P_t, t \geq 0$ satisfies the following properties:*

- (1) P_t is a self-adjoint operator;
- (2) P_t commutes with Δ , i.e. $P_t\Delta = \Delta P_t$;
- (3) $P_t P_s = P_{t+s}$ for any $t, s \geq 0$;
- (4) $P_t f(x) = \sum_{y \in V} f(y) p_t(x, y) \mu(y)$, where

$$p_t(x, y) = \sum_{i=1}^N e^{-\lambda_i t} \psi_i(x) \psi_i(y) \geq 0$$

and $\sum_{y \in V} p_t(x, y) \mu(y) = 1$. In particular, $0 \leq P_t(\chi_S) \leq 1$, where χ_S is the characteristic function of a subset $S \subset V$;

- (5) $\sum_{x \in V} P_t f(x) \mu(x) = \sum_{x \in V} f(x) \mu(x)$.

The following Bakry-Émery type gradient estimate of $P_t f$ is an important feature of the CD-inequality.

Lemma 2.11 (see e.g. [19]). *If (G, μ) satisfies $CD(-K, \infty)$ ($K \geq 0$), then we have for any function $f : V \rightarrow \mathbb{R}$,*

$$\Gamma(P_t f) \leq e^{2Kt} P_t(\Gamma(f)). \quad (18)$$

Proof. The proof in Ledoux [19] (see (5.3) there) works still for the graph setting. For any $0 \leq s \leq t$, define

$$F(s) := e^{2Ks} P_s(\Gamma(P_{t-s} f)). \quad (19)$$

Observe that $F(0) = \Gamma(P_t f)$ and $F(t) = e^{2Kt} P_t(\Gamma(f))$. Hence it remains to check the sign of the derivative $\frac{d}{ds} F(s)$. We calculate

$$\frac{d}{ds} F(s) = 2K e^{2Ks} P_s(\Gamma(P_{t-s} f)) + e^{2Ks} \Delta P_s(\Gamma(P_{t-s} f)) + e^{2Ks} P_s\left(\frac{d}{ds} \Gamma(P_{t-s} f)\right).$$

Recalling the equations (6) and (17), it is straightforward to see for any $x \in V$

$$\begin{aligned} \frac{d}{ds} \Gamma(P_{t-s}f)(x) &= \frac{1}{\mu(x)} \sum_{y, y \sim x} w_{xy} (P_{t-s}f(y) - P_{t-s}f(x)) (-\Delta P_{t-s}f(y) + \Delta P_{t-s}f(x)) \\ &= -2\Gamma(P_{t-s}f, \Delta P_{t-s}f)(x). \end{aligned}$$

Now we observe that

$$\frac{d}{ds} F(s) = 2e^{2Ks} P_s (\Gamma_2(P_{t-s}f) + K\Gamma(P_{t-s}f)) \geq 0,$$

where we used (5) and our curvature condition. This completes the proof. \square

3. EIGENVALUE RATIOS AND HIGHER ORDER SPECTRAL BOUNDS

We will use the Cheeger constant to relate the k -th with the second eigenvalue, from which we deduce an eigenvalue ratio estimate. For a given (G, μ) , recall that the expansion $\phi_{w, \mu}(S)$ of a nonempty subset S of V is defined as

$$\phi_{w, \mu}(S) := \frac{|E(S, V \setminus S)|_w}{\mu(S)},$$

where $|E(S, V \setminus S)|_w := \sum_{x \sim y, x \in S, y \notin S} w_{xy}$ and $\mu(S) := \sum_{x \in S} \mu(x)$.

Definition 3.1 (Multi-way isoperimetric constants [24, 20]). For a natural number k , the k -way isoperimetric constant of (G, μ) is defined as

$$h_k(G, \mu) := \min_{S_1, \dots, S_k} \max_{1 \leq i \leq k} \phi_{w, \mu}(S_i),$$

where the minimum is taken over all collections of k non-empty, mutually disjoint subsets S_1, \dots, S_k , i.e., all k -subpartitions of V .

Note that $h_2(G, \mu)$ coincides with the classical Cheeger constant and that $h_k(G, \mu) \leq h_{k+1}(G, \mu)$.

3.1. Eigenvalue ratio. As in [23] for the Riemannian manifold case, we need to combine the improved Cheeger inequality with a Buser type inequality. Buser's inequality is shown on graphs by Bauer et al. [4] under their exponential curvature-dimension inequality constraints. As mentioned in [4], in an unpublished manuscript [17], Klartag and Kozma proved Buser's inequality for non-normalized Laplacian with CD-inequality constraints. In our general setting it reads as below.

Theorem 3.2. *Let (G, μ) satisfy $CD(-K, \infty)$ for some $K \geq 0$. Then we have*

$$\lambda_2(G, \mu) \leq 8 \max \left\{ \sqrt{D_G^{nor} K} h_2(G, \mu), \left(\frac{e}{e-1} \right)^2 D_G^{nor} h_2^2(G, \mu) \right\}. \quad (20)$$

In particular, when (G, μ) satisfies $CD(0, \infty)$, we have

$$h_2(G, \mu) \geq \frac{e-1}{2e} \frac{1}{\sqrt{D_G^{nor}}} \sqrt{\lambda_2(G, \mu)}. \quad (21)$$

Note that inequality (21) is slightly better than the one obtained by directly inserting $K = 0$ into (20). Actually, with formulas (7), (8), Proposition 2.10 and Lemma 2.11 in hand, Theorem 3.2 follows via the same proof as for closed Riemannian manifolds in Section 5 of Ledoux [19]. For the reader's convenience, we present a proof for (21) in Appendix A.2. The dependence on D_G^{nor} comes from Lemma A.3 there, see also [4].

We also need the following improved Cheeger inequality due to Kwok et al. [18] to obtain the eigenvalue ratio estimate.

Theorem 3.3. *On (G, μ) we have for any natural number $k \geq 2$,*

$$h_2(G, \mu) \leq 10 \sqrt{2D_G^{non} k} \frac{\lambda_2(G, \mu)}{\sqrt{\lambda_k(G, \mu)}}. \quad (22)$$

Here the setting is slightly more general than that in [18]. To obtain (22), one needs to be careful about the final calculations in the proof of Proposition 3.2 in [18] (pp.16 in the full version of [18]) and the fact that $\lambda_k \leq 2D_G^{non}$.

Combining (21) and (22), we get the following eigenvalue ratio estimate.

Theorem 3.4. *For any graph (G, μ) satisfying $CD(0, \infty)$ and any natural number $k \geq 2$, we have*

$$\lambda_k(G, \mu) \leq \left(\frac{20\sqrt{2}e}{e-1} \right)^2 D_G^{non} D_G^{nor} k^2 \lambda_2(G, \mu). \quad (23)$$

We remark that this estimate does not depend on the size of the graph. Moreover, the following example shows that the order of k in the above estimate is optimal.

Example 3.5. Consider an unweighted cycle \mathcal{C}_N with $N \geq 3$ vertices. Note that \mathcal{C}_N can be considered as an abelian Cayley graph and hence satisfies $CD(0, \infty)$. Assign to it a measure μ which takes the constant value 2 on every vertex. Then the eigenvalues of the associated Laplacian are given by (see e.g. Example 1.5 in [8] or Section 7 in [22]),

$$\lambda_k(\mathcal{C}_N) = 1 - \cos \left(\frac{2\pi}{N} \left\lfloor \frac{k}{2} \right\rfloor \right), \quad k = 1, 2, \dots, N.$$

Observe that we have

$$\lim_{N \rightarrow \infty} \frac{\lambda_k(\mathcal{C}_N)}{\lambda_2(\mathcal{C}_N)} = \left\lfloor \frac{k}{2} \right\rfloor^2.$$

The dependence on the term $D_G^{non} D_G^{nor}$ is also necessary in the estimate (23). This can be concluded from the following examples.

Example 3.6. Let us revisit the triangle graph (Δ_{xyz}, μ) in Example 2.3. Consider the special case that $A = B = C = 1$ and $a = c$. Suppose $b \geq a$. Then this graph satisfies $CD(0, \infty)$. The eigenvalues of the non-normalized Laplacian are

$$\lambda_1 = 0 < \lambda_2 = 3a \leq \lambda_3 = a + 2b.$$

Note further that $D_G^{non} D_G^{nor} = (a + b)/a$. Therefore, we have

$$\frac{1}{3} D_G^{non} D_G^{nor} \leq \frac{\lambda_3(\Delta_{xyz})}{\lambda_2(\Delta_{xyz})} \leq \frac{2}{3} D_G^{non} D_G^{nor}. \quad (24)$$

We give another example which works for the eigenvalue ratios of both non-normalized and normalized Laplacians.

Example 3.7. Consider the tetrahedron graph (T_4, μ) in Example 2.4 with the assumption that $b \geq a = c$. Recall $\mu = A$ is a constant measure. Then the eigenvalues of the μ -Laplacian are

$$\lambda_1 = 0 < \lambda_2 = \frac{4a}{A} \leq \lambda_3 = \lambda_4 = \frac{2a + 2b}{A}.$$

Moreover, we have $D_G^{non} D_G^{nor} = (2a + b)/a$. Hence we obtain

$$\frac{1}{4} D_G^{non} D_G^{nor} \leq \frac{\lambda_3(T_4)}{\lambda_2(T_4)} \leq D_G^{non} D_G^{nor}. \quad (25)$$

The following example shows that we cannot expect that the eigenvalue ratio estimate (23) remains valid if a graph (G, μ) possesses a small portion of vertices not satisfying $CD(0, \infty)$.

Example 3.8. Consider a sequence of dumbbell graphs $\{G_N\}_{N=3}^\infty$. Given two copies of complete graphs over N vertices, \mathcal{K}_N and \mathcal{K}'_N , G_N is the graph obtained via connecting them by a new edge $e = (y_0, y'_0)$ as shown in Figure 3. It was shown in [16] that the complete graph \mathcal{K}_N with normalized

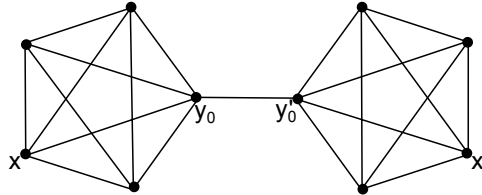


FIGURE 3. The dumbbell graph G_5

Laplacian satisfies $CD\left(\frac{N+2}{2(N-1)}, \infty\right)$. Modifying the calculation in the proof of this fact in [16], we obtain the following results.

- With the normalized Laplacian, G_N satisfies $CD\left(\frac{1}{2}, \infty\right)$ at every vertex which is not y_0, y'_0 . At y_0, y'_0 , $CD(0, \infty)$ does not hold when $N \geq 3$.

- With the non-normalized Laplacian, G_N satisfies $CD\left(\frac{N}{2}, \infty\right)$ at every vertex which is not y_0, y'_0 . At y_0, y'_0 , $CD(0, \infty)$ does not hold when $N \geq 3$.

We present the calculations in Appendix A.3. With only 2 of $2N$ vertices violating the curvature condition, the eigenvalue ratio estimate (23) does not hold any more. Indeed, for the normalized Laplacian, we observe by Cheeger's inequality that $\lambda_2(G_N) \leq \frac{2}{N(N-1)+1}$. Recall that the spectrum of a complete graph \mathcal{K}_N is the simple eigenvalue 0 and the eigenvalue $\frac{N}{N-1}$ with multiplicity $N-1$. By an interlacing theorem for edge-deleting due to Chen et al. [5], we conclude that $\lambda_4(G_N) \geq \frac{N}{N-1}$. Therefore we have

$$\frac{\lambda_4(G_N)}{\lambda_2(G_N)} \geq \frac{1}{2}N^2.$$

Since in this case $D_G^{non}D_G^{nor} = N$, (23) does not hold when N is large. Similar arguments show also for the non-normalized Laplacian that (23) is no longer true for all N . (The interlacing theorem for non-normalized Laplacian is well-known, see e.g. [14]).

Remark 3.9. Replacing the sequence of complete graphs \mathcal{K}_N above by a sequence of expanders, we obtain graphs of *bounded degree* violating the curvature condition and for which (23) does not hold.

3.2. Higher-order Buser inequality. A higher-order Buser inequality was first established by Funano [13] in the Riemannian setting and then improved in [23]. The following inequality seems to be the first higher-order Buser type inequality in the graph setting.

Theorem 3.10. *For any graph (G, μ) satisfying $CD(0, \infty)$ and any natural number k , we have*

$$h_k(G, \mu) \geq h_2(G, \mu) \geq \frac{(e-1)^2}{40\sqrt{2}e^2} \frac{1}{D_G^{nor} \sqrt{D_G^{non}}} \frac{1}{k} \sqrt{\lambda_k(G, \mu)}. \quad (26)$$

Proof. The first inequality is given by the monotonicity of the multi-way isoperimetric constant. The second inequality follows from Buser's inequality (21) and Theorem 3.4. \square

Remark 3.11. The higher-order Cheeger inequality of Lee-Oveis Gharan-Trevisan [20] reads as

$$h_k(G, \mu) \leq C \sqrt{D_G^{non}} k^2 \sqrt{\lambda_k(G, \mu)}, \quad (27)$$

where C is an universal constant. (For the generalization into our setting, one needs to slightly modify the calculation for $\mathbb{E}\left(\sum_{i=1}^m w(E(\hat{S}_i, \bar{S}_i))\right)$ in Lemma 4.7 of [20].) Hence, for a graph (G, μ) satisfying $CD(0, \infty)$ and with bounded degree, $h_k(G, \mu)$ and $\sqrt{\lambda_k(G, \mu)}$ are equivalent up to polynomials of k .

Remark 3.12. In [4], Bauer et al. proved for a graph (G, μ) satisfying the exponential curvature-dimension inequality $CDE(0, n)$ (see Definition 3.9 in [4]) and for a fixed $0 < \alpha < 1$ that there exists a constant $C(\alpha)$, depending only on α , such that

$$\lambda_2(G, \mu) \leq C(\alpha) D_G^{nor} n h_2(G, \mu)^2. \quad (28)$$

That is, they obtain a dimension-dependent Buser inequality. Our approach also applies to their setting. In particular, we obtain the following eigenvalue ratio estimate and higher-order Buser inequality under the condition $CDE(0, n)$,

$$\lambda_k(G, \mu) \leq C_1(\alpha) D_G^{nor} D_G^{non} n k^2 \lambda_2(G, \mu), \quad (29)$$

$$h_k(G, \mu) \geq h_2(G, \mu) \geq C_2(\alpha) \frac{1}{D_G^{nor} \sqrt{D_G^{non}}} \frac{1}{n k} \sqrt{\lambda_k(G, \mu)}, \quad (30)$$

where $C_1(\alpha)$, $C_2(\alpha)$ are constants depending only on α .

3.3. Higher-order Alon-Milman inequalities. In this subsection, we derive the higher-order counterparts of the Alon-Milman [1] concentration inequality and the eigenvalue-diameter estimate for graphs satisfying $CD(0, \infty)$. The higher order eigenvalue-diameter estimate can be considered as an analogue of the celebrated Cheng estimate (Corollary 2.2 in [6]) in the setting of compact Riemannian manifolds with nonnegative Ricci curvature.

Given a graph $(G = (V, E, w), \mu)$ and two disjoint subset S_1, S_2 of V , we introduce

$$s_1 := \frac{\mu(S_1)}{\mu(V)} \text{ and } s_2 := \frac{\mu(S_2)}{\mu(V)}.$$

Then we have the following result.

Theorem 3.13. *Let (G, μ) satisfy $CD(0, \infty)$. Assume that the distance $\text{dist}(S_1, S_2)$ between two subsets S_1 and S_2 is larger than $\rho \geq 1$. Then we have for any $k \geq 2$,*

$$s_2 \leq (1 - s_1) \exp \left\{ -\ln(1 + 2s_1) \left[\frac{e - 1}{20\sqrt{2}e D_G^{non} k} \sqrt{\frac{\lambda_k(G, \mu)}{D_G^{nor}}} \rho \right] \right\}. \quad (31)$$

This inequality is closely related to constructing a concentrated family of graphs (see Definition 1.2 in [1]). It can also be considered as an asymptotic isoperimetric inequality since it provides an estimate for the maximal possible μ -volume of vertices of distance larger than ρ from a set possessing half of the whole μ -volume.

Proof. We first need to extend the concentration inequality in terms of the smallest non-zero eigenvalue of the non-normalized Laplacian of Alon-Milman (see Theorem 2.6 in [1]) to our present setting. We outline the critical steps here. As a finer version of the easier part of Cheeger's inequality, we have

$$\lambda_2(G, \mu) \mu(V) \leq \frac{1}{\text{dist}(S_1, S_2)^2} \left(\frac{1}{s_1} + \frac{1}{s_2} \right) (|E|_w - |E_{S_1}|_w - |E_{S_2}|_w), \quad (32)$$

where we use E_{S_i} for the set of edges with both ends in S_i , $i = 1, 2$ and $|\cdot|_w$ for the sum of weights of edges in the corresponding set. Observe that when $\text{dist}(S_1, S_2) > 1$, every edge in $E \setminus (E_{S_1} \cup E_{S_2})$ is incident with at least one of the vertices in $V \setminus (S_1 \cup S_2)$. Hence we have

$$\begin{aligned} |E|_w - |E_{S_1}|_w - |E_{S_2}|_w &\leq \sum_{x \in V \setminus (S_1 \cup S_2)} \sum_{y, y \sim x} w_{xy} \\ &\leq D_G^{\text{non}}(\mu(V) - \mu(S_1) - \mu(S_2)) = D_G^{\text{non}}\mu(V)(1 - s_1 - s_2). \end{aligned}$$

Combining this with (32), we obtain in the case $\text{dist}(S_1, S_2) > 1$ that

$$s_2 \leq \frac{1 - s_1}{1 + (\lambda_2(G, \mu)/D_G^{\text{non}})s_1 \text{dist}(S_1, S_2)^2}. \quad (33)$$

Applying of the above inequality iteratively, we arrive at (see details in [1])

$$s_2 \leq (1 - s_1) \exp \left\{ -\ln(1 + 2s_1) \left\lfloor \frac{\lambda_2(G, \mu)}{2D_G^{\text{non}}} \rho \right\rfloor \right\}. \quad (34)$$

Now (31) follows from (34) and Theorem 3.4. \square

Theorem 3.14. *If (G, μ) satisfies $CD(0, \infty)$ then we have for the diameter of G and any $k \geq 2$*

$$\text{diam}(G) \leq 2 \left\lfloor \frac{40e}{e-1} D_G^{\text{non}} \sqrt{D_G^{\text{nor}}} \frac{k}{\sqrt{\lambda_k(G, \mu)}} \log_2 \frac{\mu(V)}{\min_{x \in V} \mu(x)} \right\rfloor. \quad (35)$$

Proof. We can either derive this theorem directly from Theorem 3.13 or first extend the Alon-Milman eigenvalue-diameter estimate to our present setting as

$$\text{diam}(G) \leq 2 \left\lfloor \sqrt{\frac{2D_G^{\text{non}}}{\lambda_2(G, \mu)}} \log_2 \frac{\mu(V)}{\min_{x \in V} \mu(x)} \right\rfloor. \quad (36)$$

For this extension, we only need to be aware of the fact that $\mu(S) < \min_{x \in V} \mu(x)$ implies $S = \emptyset$. Then combining (36) with Theorem 3.4 implies (35). \square

Remark 3.15. Note that there are various further developments in connection with Alon-Milman's estimate (36), see, e.g., the work of Chung [7], Mohar [26], Chung-Grigor'yan-Yau [9] and Houdré-Tetali [15]. In principle, the estimate (35) can be improved accordingly.

4. RATIOS OF MULTI-WAY ISOPERIMETRIC CONSTANTS

Theorem 4.1. *There exists a universal constant C such that for any graph (G, μ) satisfying $CD(0, \infty)$ and for any natural number $k \geq 2$, we have*

$$h_k(G, \mu) \leq CD_G^{\text{non}} D_G^{\text{nor}} k \sqrt{\log k} h_2(G, \mu). \quad (37)$$

Proof. For this estimate, we need the following result of Lee-Oveis Gharan-Trevisan [20] (Theorem 1.2 there),

$$h_k(G, \mu) \leq C_1 \sqrt{D_G^{non} \log k \lambda_{2k}}, \quad (38)$$

where C_1 is a universal constant. Then (37) follows from applying Theorem 3.4 and Buser inequality (21) in turn. \square

In fact, Lee-Oveis Gharan-Trevisan removed the dependence on k in the estimate (38) in the case when more information about the geometry of the graph (G, μ) is known. When G has genus at most $g \geq 1$ (i.e. G can be embedded into an orientable surface of genus at most g without edge crossings), they prove that there exists a universal constant C'_1 such that

$$h_k(G, \mu) \leq C'_1 \log(g+1) \sqrt{D_G^{non} \lambda_{2k}}. \quad (39)$$

Therefore we have the following result.

Corollary 4.2. *There exists a universal constant C such that if (G, μ) satisfies $CD(0, \infty)$, then for any $k \geq 2$,*

$$h_k(G, \mu) \leq CD_G^{non} D_G^{nor} \log(g_G + 1) k h_2(G, \mu), \quad (40)$$

where $g_G \geq 1$ is an upper bound of the genus of G .

Remark 4.3. The order of k in (40) is optimal. This follows from the example of unweighted cycles \mathcal{C}_N (which are planar) with the same measure μ as in Example 3.5, since we have (see e.g. Proposition 7.3 in [22]).

$$h_k(\mathcal{C}_N) = \frac{1}{\lfloor \frac{N}{k} \rfloor}, \quad \text{for } 2 \leq k \leq N.$$

The dependence on $D_G^{non} D_G^{nor}$ of the ratio estimate is also necessary. This follows from the following example analyzed in Mimura [25].

Example 4.4. Consider the Cartesian product graph $G_{N,2}$ of the unweighted complete graphs \mathcal{K}_N and \mathcal{K}_2 . Assign the measure $\mu = \mathbf{1}$ to it. Since complete graphs satisfy $CD(0, \infty)$ (see [16]), we know by Theorem 2.5 that $G_{N,2}$ satisfies $CD(0, \infty)$. It is straightforward to see that $h_2(G_{N,2}) \leq 1$. Observe that we can partition $G_{N,2}$ into two induced subgraphs \mathcal{K}_N and \mathcal{K}'_N . By Lemma 1 of Tanaka [27] (see also [25]), we have

$$h_3(G_{N,2}) \geq h_2(\mathcal{K}_N) = \frac{N}{2}.$$

(Note that Tanaka's lemma was stated for the constants $\{\mathfrak{h}_k(G)\}$ defined below. One can check that it also works for $\{h_k(G)\}$ here.) Therefore, we obtain

$$\frac{h_3(G_{N,2})}{h_2(G_{N,2})} \geq \frac{N}{2} = \frac{1}{2} d_G. \quad (41)$$

This shows the necessity of the dependence on the term $D_G^{non} D_G^{nor} = d_G$. Note (41) also holds for the normalized measure μ . We comment that one can also analyse the eigenvalues of this example to show the necessity of the

dependence on the degree in (23) (see also [25]) using an interlacing theorem or Lemma 6 of [27].

Now we restrict our considerations to the setting $w \equiv 1$ and $\mu = \mathbf{1}$, that is, $G = (V, E)$ is now an unweighted graph with non-normalized Laplacian.

Recently, the concept of multi-way expanders was defined and studied in Tanaka [27] and Mimura [25]. We denote $\mathfrak{h}_k(G)$ to be the following larger k -way isoperimetric constant (compare with Definition 3.1)

$$\mathfrak{h}_k(G) := \min_{S_1, \dots, S_k} \max_{1 \leq i \leq k} \phi_{1,1}(S_i), \quad (42)$$

where the minimum is taken over all partitions of V , i.e. $V = \bigsqcup_{i=1}^k S_i$, $S_i \neq \emptyset$ for all i .

Definition 4.5 (Multi-way expanders [27, 25]). Let $k \geq 2$ be a natural number. A sequence of finite graphs $\{G_m = (V_m, E_m)\}_{m \in \mathbb{N}}$ is called a sequence of k -way expanders if we have (i) $\sup_m d_{G_m} < \infty$; (ii) $\lim_{m \rightarrow \infty} |V_m| = \infty$; (iii) $\inf_m \mathfrak{h}_k(G_m) > 0$.

Observe that 2-way expander families coincide with classical families of expanders. In general, the property of being $(k+1)$ -way expanders is strictly weaker than being k -way expanders (see [25]). However, Mimura [25] proved that the concepts of k -way expanders for all $k \geq 2$ are equivalent within the class of finite, connected, vertex transitive graphs.

As a consequence of Theorem 4.1, we have

Corollary 4.6. *For the class of finite connected graphs satisfying $CD(0, \infty)$, the concepts of k -way expanders for all $k \geq 2$ are equivalent.*

Proof. Using the relation (see Theorem 3.8 in [20], [25])

$$h_k(G) \leq \mathfrak{h}_k(G) \leq k h_k(G), \quad (43)$$

and employing (37) yields

$$\mathfrak{h}_k(G) \leq C d_G k^2 \sqrt{\log k} \mathfrak{h}_2(G). \quad (44)$$

Hence, when $d_G < \infty$, $\inf_m \mathfrak{h}_k(G_m) > 0$ implies $\inf_m \mathfrak{h}_2(G_m) > 0$. This completes the proof. \square

Abelian Cayley graphs lie in the intersection of the class of vertex transitive graphs and the class of graphs satisfying $CD(0, \infty)$. It is well known that there are no expanders in the class of abelian Cayley graphs (see Alon-Roichman [2]). Moreover, Friedman-Murty-Tillich [12] proved an explicit upper estimate for λ_2 which implies this fact. Therefore, we also obtain the following explicit upper estimate for λ_k implying the nonexistence of sequences of multi-way expanders in this class.

Corollary 4.7. *For any abelian Cayley graphs with degree d and of vertex size N , there exists a universal constant C such that for any $k \geq 2$,*

$$\lambda_k \leq C k^2 d^2 N^{-\frac{4}{d}}. \quad (45)$$

As indicated, this is a direct consequence of Theorem 3.4 and the estimate $\lambda_2 \leq CdN^{-\frac{4}{d}}$ in [12]. Therefore, it is natural to ask the following question.

Question 4.8. Does there exist a sequence of expanders satisfying $CD(0, \infty)$?

We are inclined to a negative answer. For example, the nonexistence of expander families satisfying $CD(0, \infty)$ would follow if one could prove that every graph of vertex degree at most d and satisfying $CD(0, \infty)$ possesses polynomial volume growth with degree depending only on d .

APPENDIX

A.1. Curvature matrix of the triangle and tetrahedron graphs. The curvature matrix $\Gamma_2(x)$ for the graph (\triangle_{xyz}, μ) in Figure 1 is

$$\frac{1}{4C} \begin{pmatrix} \frac{3a^2}{B} + \frac{3b^2}{A} + \frac{(a+b)^2}{C} & \frac{bc}{A} - \frac{a(3a+c)}{B} - \frac{a(a+b)}{C} & \frac{ac}{B} - \frac{b(3b+c)}{A} - \frac{b(a+b)}{C} \\ \frac{bc}{A} - \frac{a(3a+c)}{B} - \frac{a(a+b)}{C} & \frac{bc}{A} + \frac{3a(a+c)}{B} + \frac{a(a-b)}{C} & \frac{2ab}{C} - \frac{2ac}{B} - \frac{2bc}{A} \\ \frac{ac}{B} - \frac{b(3b+c)}{A} - \frac{b(a+b)}{C} & \frac{2ab}{C} - \frac{2ac}{B} - \frac{2bc}{A} & \frac{3b(b+c)}{A} + \frac{b(b-a)}{C} + \frac{ac}{B} \end{pmatrix}.$$

Let us have a closer look at the special case that $A = B = C = 1$ and $a = c$. Then the matrix $4\Gamma_2(x) = 4\Gamma_2(z)$ reduces to

$$\begin{pmatrix} 4a^2 + 2ab + 4b^2 & -5a^2 & a^2 - 2ab - 4b^2 \\ -5a^2 & 7a^2 & -2a^2 \\ a^2 - 2ab - 4b^2 & -2a^2 & a^2 + 2ab + 4b^2 \end{pmatrix},$$

and the matrix $4\Gamma_2(y)$ is

$$\begin{pmatrix} 10a^2 & -5a^2 & -5a^2 \\ -5a^2 & 3a^2 + 4ab & 2a^2 - 4ab \\ -5a^2 & 2a^2 - 4ab & 3a^2 + 4ab \end{pmatrix}.$$

Observe that when $b \geq a/2$, the above two matrices are both diagonally dominant and hence positive-semidefinite. In fact they are always positive-semidefinite for any $a, b \geq 0$.

The matrix $4A^2\Gamma_2(x)$ for the tetrahedron graph (T_4, μ) in Figure 2 is given by

$$\begin{pmatrix} 2ab+2ac+2bc+4a^2+4b^2+4c^2 & -2ab+2ac-2bc-4b^2 & 2ab-2ac-2bc-4c^2 & -2ab-2ac+2bc-4a^2 \\ -2ab+2ac-2bc-4b^2 & 2ac+2ab+2bc+4b^2 & -2ab-2ac+2bc & -2bc-2ac+2ab \\ 2ab-2ac-2bc-4c^2 & -2ab-2ac+2bc & 2ab+2ac+2bc+4c^2 & -2ab+2ac-2bc \\ -2ab-2ac+2bc-4a^2 & -2bc-2ac+2ab & -2ab+2ac-2bc & 2ab+2ac+2bc+4a^2 \end{pmatrix}.$$

This is a positive-semidefinite matrix.

A.2. Proof of Buser inequality. In this subsection, we present a proof of Buser's inequality (21), following closely Ledoux [19] (see also [4, 17]). The following Lemma can be considered as a reverse Poincaré inequality.

Lemma A.1. *Assume that (G, μ) satisfies $CD(0, \infty)$. Then, we have for any function $f : V \rightarrow \mathbb{R}$, any $t \geq 0$, and any $x \in V$,*

$$P_t(f^2)(x) - (P_t f)^2(x) \geq 2t\Gamma(P_t f)(x). \quad (46)$$

Proof. For $0 \leq s \leq t$, set $G(s) = P_s((P_{t-s}f)^2)$. Then we have $G(0) = (P_t f)^2$, $G(t) = P_t(f^2)$. Using the gradient estimate in Lemma 2.11, we have

$$\begin{aligned} \frac{d}{ds}G(s) &= \Delta P_s((P_{t-s}f)^2) + P_s(-2P_{t-s}f \Delta P_{t-s}f) \\ &= 2P_s(\Gamma(P_{t-s}f)) \geq 2\Gamma(P_t f). \end{aligned}$$

Now we arrive at

$$G(t) - G(0) = \int_0^t \frac{d}{ds}G(s)ds \geq 2\Gamma(P_t f) \int_0^t ds = 2t\Gamma(P_t f).$$

This completes the proof. \square

Define the l_p norm of a function $f : V \rightarrow \mathbb{R}$ as $\|f\|_p := (\sum_{x \in V} \mu(x)|f(x)|^p)^{\frac{1}{p}}$. We have the following direct corollary.

Corollary A.2. *Assume that (G, μ) satisfies $CD(0, \infty)$. Then, we have for any function $f : V \rightarrow \mathbb{R}$ and any $t \geq 0$,*

$$\|f - P_t f\|_1 \leq \sqrt{2t} \|\sqrt{\Gamma(f)}\|_1.$$

Proof. Note first that, by Lemma A.1 and Proposition 2.10 (4), we have

$$\|\sqrt{\Gamma(P_t f)}\|_\infty \leq \frac{1}{\sqrt{2t}} \|f\|_\infty. \quad (47)$$

Let $g : V \rightarrow \mathbb{R}$ be $g(x) := \text{sgn}(f(x) - P_t f(x))$. Then we obtain

$$\begin{aligned} \|f - P_t f\|_1 &= \sum_{x \in V} \mu(x)(f(x) - P_t f(x))g(x) = - \sum_{x \in V} \mu(x) \int_0^t \Delta P_s f(x) ds g(x) \\ &= - \int_0^t \sum_{x \in V} \mu(x) P_s g(x) \Delta f(x) ds = \int_0^t \sum_{x \in V} \mu(x) \Gamma(P_s g(x), f(x)) ds \\ &\leq \int_0^t \sum_{x \in V} \mu(x) \sqrt{\Gamma(P_s g)(x)} \sqrt{\Gamma(f)(x)} ds \\ &\leq \|\sqrt{\Gamma(f)}\|_1 \int_0^t \|\sqrt{\Gamma(P_s g)}\|_\infty ds \leq \|\sqrt{\Gamma(f)}\|_1 \|g\|_\infty \int_0^t \frac{1}{\sqrt{2s}} ds \\ &\leq \sqrt{2t} \|\sqrt{\Gamma(f)}\|_1, \end{aligned}$$

where we used Proposition 2.10 (1-2), (7), (8) and (47). \square

Recall that χ_S denotes the characteristic function of $S \subset V$.

Lemma A.3. *We have*

$$\|\sqrt{\Gamma(\chi_S)}\|_1 \leq \sqrt{2D_G^{\text{nor}}} E(S, V \setminus S). \quad (48)$$

Proof. This follows from the direct calculation given here:

$$\begin{aligned}
\|\sqrt{\Gamma(\chi_S)}\|_1 &= \sum_{x \in V} \mu(x) \sqrt{\frac{1}{2\mu(x)} \sum_{y, y \sim x} w_{xy} (\chi_S(x) - \chi_S(y))^2} \\
&\leq \sum_{x \in V} \sqrt{\frac{\mu(x)}{2}} \sum_{y, y \sim x} \sqrt{w_{xy}} |\chi_S(x) - \chi_S(y)| \\
&\leq \sqrt{\frac{D_G^{nor}}{2}} \sum_{x \in V} \sum_{y, y \sim x} w_{xy} |\chi_S(x) - \chi_S(y)| \\
&= \sqrt{2D_G^{nor}} E(S, V \setminus S).
\end{aligned}$$

□

Proof of (21). Using Corollary A.2 and Lemma A.3, we have

$$\begin{aligned}
\sqrt{2t} \|\sqrt{\Gamma(\chi_S)}\|_1 &\leq 2\sqrt{D_G^{nor} t} E(S, V \setminus S) \\
&\geq \sum_{x \in V} \mu(x) |\chi_S(x) - P_t \chi_S(x)| = \sum_{x \in S} (1 - P_t \chi_S(x)) + \sum_{x \in V \setminus S} \mu(x) P_t \chi_S(x) \\
&= 2\mu(S) - 2 \sum_{x \in S} P_t(\chi_S)(x) \mu(x),
\end{aligned} \tag{49}$$

where we used Proposition 2.10 (4-5).

Now let $\{\alpha_i\}_{i=1}^N$ be N constants such that $\chi_S = \sum_{i=1}^N \alpha_i \psi_i$, where $\{\psi_i\}_{i=1}^N$ are the orthonormal basis of $l^2(V, \mu)$ given by eigenfunctions. Then we have $\|\chi_S\|_2^2 = \sum_{i=1}^N \alpha_i^2 = \mu(S)$ and

$$\alpha_1 = (\chi_S, \psi_1) = \sum_{x \in V} \mu(x) \chi_S(x) \frac{1}{\sqrt{\mu(V)}} = \frac{\mu(S)}{\sqrt{\mu(V)}}.$$

Now we have by Proposition 2.10 (4)

$$\begin{aligned}
\sum_{x \in S} P_t(\chi_S)(x) \mu(x) &= \sum_{x \in V} (P_{\frac{t}{2}} \chi_S)^2(x) \mu(x) \\
&= \sum_{x \in V} \left(\sum_{i=1}^N e^{-\frac{\lambda_i t}{2}} \alpha_i \psi_i \right)^2 \mu(x) = \sum_{i=1}^N e^{-\lambda_i t} \alpha_i^2 \leq e^{-\lambda_2 t} \sum_{i=2}^N \alpha_i^2 + \alpha_0^2 \\
&= e^{-\lambda_2 t} \left(\mu(S) - \frac{\mu(S)^2}{\mu(V)} \right) + \frac{\mu(S)^2}{\mu(V)}.
\end{aligned}$$

Inserting the above estimate into (49), we arrive at

$$2\sqrt{D_G^{nor} t} E(S, V \setminus S) \geq 2 \left(\mu(S) - \frac{\mu(S)^2}{\mu(V)} \right) (1 - e^{-\lambda_2 t}). \tag{50}$$

Taking $t = \frac{1}{\lambda_2}$, we obtain for those S with $\mu(S) \leq \frac{1}{2}\mu(V)$,

$$2\sqrt{\frac{D_G^{nor}}{\lambda_2}} E(S, V \setminus S) \geq \mu(S)(1 - e^{-1}).$$

This completes the proof. \square

A.3. CD-inequalities of dumbbell graphs. In this subsection we present the calculations for the CD-inequalities of dumbbell graphs G_N claimed in Example 3.8. They are modified from that of Proposition 3 in [16].

A general formula representing $\Gamma_2(f)$ is given by

$$\begin{aligned} \Gamma_2(f)(x) = & Hf(x) + \frac{1}{2}(\Delta f(x))^2 - \frac{1}{2} \frac{\sum_{y, y \sim x} w_{xy}}{\mu(x)} \Gamma(f)(x) \\ & - \frac{1}{4} \frac{1}{\mu(x)} \sum_{y, y \sim x} w_{xy} (f(y) - f(x))^2 \frac{\sum_{z, z \sim y} w_{yz}}{\mu(y)}, \end{aligned} \quad (51)$$

where

$$Hf(x) := \frac{1}{4} \frac{1}{\mu(x)} \sum_{y, y \sim x} \frac{w_{xy}}{\mu(y)} \sum_{z, z \sim y} w_{yz} (f(x) - 2f(y) + f(z))^2.$$

This is an extension of (2.9) in [16] to our general setting (G, μ) .

Let us first consider the case of the unweighted normalized Laplacian. Let x be a vertex of G_N which is different from y_0 or y'_0 (see Figure 3). First observe that

$$Hf(x) \geq \frac{1}{4N(N-1)} \sum_{y, y \sim x} \sum_{\substack{z, z \sim y \\ z \neq y'_0}} (f(x) - 2f(y) + f(z))^2.$$

Now our calculations reduce to the complete graph \mathcal{K}_N itself. Note that when $y, z \neq x$,

$$\begin{aligned} & (f(x) - 2f(y) + f(z))^2 + (f(x) - 2f(z) + f(y))^2 \\ & = (f(x) - f(y))^2 + (f(x) - f(z))^2 + 4(f(y) - f(z))^2. \end{aligned}$$

Then we have

$$Hf(x) \geq \frac{N+2}{2N} \Gamma(f)(x) + \frac{1}{N(N-1)} \sum_{(y,z)} (f(y) - f(z))^2,$$

where the second summation is over all unordered pair of neighbors of x . By (51), we arrive at

$$\Gamma_2(f)(x) \geq \frac{2-N}{2N} \Gamma(f)(x) + \frac{1}{2} (\Delta f(x))^2 + \frac{1}{N(N-1)} \sum_{(y,z)} (f(y) - f(z))^2.$$

The last two terms above can be further manipulated as follows,

$$\begin{aligned}
& \frac{1}{2(N-1)^2} \left(\sum_{y, y \sim x} (f(y) - f(x)) \right)^2 + \frac{1}{N(N-1)} \sum_{(y,z)} (f(y) - f(z))^2 \\
& \geq \frac{1}{N(N-1)} \left[\frac{1}{2} \sum_{y, y \sim x} (f(y) - f(x))^2 - \sum_{(y,z)} (f(y) - f(x))(f(z) - f(x)) \right. \\
& \quad \left. + \sum_{(y,z)} ((f(y) - f(x))^2 + (f(z) - f(x))^2) \right] \\
& = \frac{1}{N(N-1)} \left[\left(\frac{1}{2} + \frac{N-2}{2} \right) \sum_{y, y \sim x} (f(y) - f(x))^2 + \frac{1}{2} \sum_{(y,z)} (f(y) - f(z))^2 \right] \\
& \geq \frac{N-1}{N} \Gamma(f)(x).
\end{aligned}$$

Therefore we have $\Gamma_2(f)(x) \geq \frac{1}{2} \Gamma(f)(x)$. That is, G_N satisfies $CD(\frac{1}{2}, \infty)$ at any vertex $x \neq y_0, y'_0$.

Remark A.4. We note that this CD-inequality at vertex x still holds even if we attach different graphs to every vertex in \mathcal{K}_N other than x via single edges.

At y_0 , $CD(0, \infty)$ does not hold. Let f_0 be the function taking the value 1 at y'_0 , 2 at all other vertices in \mathcal{K}'_N , and 0 at all vertices in \mathcal{K}_N . Then one can check by (51) that

$$\Gamma_2(f_0)(y_0) = \frac{3-N}{2N^2} < 0, \text{ if } N \geq 4.$$

In the case $N = 3$, we can use another function g_0 taking the value 1 at y_0 , -1 at all other vertices in \mathcal{K}_3 , 4 at y'_0 , and 7 at other vertices in \mathcal{K}'_3 . One can then check directly that $\Gamma_2(g_0)(y_0) = -\frac{1}{9} < 0$.

For the case of the unweighted non-normalized Laplacian, the calculations are similar. Note in this case at $x \neq y_0, y'_0$, we have

$$\begin{aligned}
\Gamma_2(f)(x) &= Hf(x) + \frac{1}{2} (\Delta f(x))^2 - \frac{d_x}{2} \Gamma(f)(x) - \frac{1}{4} \sum_{y, y \sim x} (f(y) - f(x))^2 d_y \\
&\geq Hf(x) + \frac{1}{2} (\Delta f(x))^2 - N \Gamma(f)(x).
\end{aligned}$$

Carrying out the calculation in the same way as in the normalized case we finally conclude $\Gamma_2(f)(x) \geq \frac{N}{2} \Gamma(f)(x)$. The arguments for CD-inequalities at y_0, y'_0 can be done with the same special functions as in the normalized case.

ACKNOWLEDGEMENTS

We thank Frank Bauer for valuable communications about Buser's inequality on graphs in [4] and [17]. This work was supported by the EPSRC Grant EP/K016687/1 "Topology, Geometry and Laplacians of Simplicial Complexes".

REFERENCES

- [1] N. Alon and V. Milman, λ_1 , isoperimetric inequalities for graphs, and superconcentrators, *J. Combin. Theory Ser. B* 38 (1985), no. 1, 73-88.
- [2] N. Alon and Y. Roichman, Random Cayley graphs and expanders, *Random Structures Algorithms* 5 (1994), no. 2, 271-284.
- [3] D. Bakry and M. Émery, Diffusions hypercontractives (French) [Hypercontractive diffusions], *Séminaire de probabilités, XIX, 1983/84, Lecture Notes in Math.* 1123, J. Azéma and M. Yor (Editors), Springer, Berlin, 1985, pp. 177-206.
- [4] F. Bauer, P. Horn, Y. Lin, G. Lippner, D. Mangoubi and S.-T. Yau, Li-Yau inequality on graphs, arXiv:1306.2561, September 2013.
- [5] G. Chen, G. Davis, F. Hall, Z. Li, K. Patel and M. Stewart, An interlacing result on normalized Laplacians, *SIAM J. Discrete Math.* 18 (2004), no. 2, 353-361.
- [6] S.-Y. Cheng, Eigenvalue comparison theorems and its geometric applications, *Math. Z.* 143 (1975), no. 3, 289-297.
- [7] F. R. K. Chung, Diameters and eigenvalues, *J. Amer. Math. Soc.* 2 (1989), no. 2, 187-196.
- [8] F. R. K. Chung, Spectral graph theory, CBMS Regional Conference Series in Mathematics 92, American Mathematical Society, Providence, RI, 1997.
- [9] F. R. K. Chung, A. Grigor'yan and S.-T. Yau, Eigenvalues and diameters for manifolds and graphs, Tsing Hua lectures on geometry & analysis (Hsinchu, 1990-1991), 79-105, Int. Press, Cambridge, MA, 1997.
- [10] F. R. K. Chung, Y. Lin and S.-T. Yau, Harnack inequalities for graphs with non-negative Ricci curvature, *J. Math. Anal. Appl.* 415 (2014), 25-32.
- [11] F. R. K. Chung, S.-T. Yau, Logarithmic Harnack inequalities, *Math. Res. Lett.* 3 (1996), no. 6, 793-812.
- [12] J. Friedman, R. Murty and J.-P. Tillich, Spectral estimates for abelian Cayley graphs, *J. Combin. Theory Ser. B* 96 (2006), no. 1, 111-121.
- [13] K. Funano, Eigenvalues of Laplacian and multi-way isoperimetric constants on weighted Riemannian manifolds, arXiv:1307.3919v1, July 2013.
- [14] J. van den Heuvel, Hamilton cycles and eigenvalues of graphs, *Linear Algebra Appl.* 226/228 (1995), 723-730.
- [15] C. Houdré and P. Tetali, Concentration of measure for products of Markov kernels and graph products via functional inequalities, *Combin. Probab. Comput.* 10 (2001), no. 1, 1-28.
- [16] J. Jost and S. Liu, Ollivier's Ricci curvature, local clustering and curvature-dimension inequalities on graphs, *Discrete Comput. Geom.* 51 (2014), no. 2, 300-322.
- [17] B. Klartag and G. Kozma, unpublished manuscript.
- [18] T.-C. Kwok, L.-C. Lau, Y.-T. Lee, S. Oveis Gharan and L. Trevisan, Improved Cheeger's inequality: Analysis of spectral partitioning algorithms through higher order spectral gap, STOC'13-Proceedings of the 2013 ACM Symposium on Theory of Computing, 11-20, ACM, New York, 2013.
- [19] M. Ledoux, Spectral gap, logarithmic Sobolev constant, and geometric bounds, *Surveys in differential geometry, Vol. IX*, 219-240, *Surv. Differ. Geom.*, IX, Int. Press, Somerville, MA, 2004.

- [20] J. R. Lee, S. Oveis Gharan and L. Trevisan, Multi-way spectral partitioning and higher-order Cheeger inequalities, STOC'12-Proceedings of the 2012 ACM Symposium on Theory of Computing, 1117-1130, ACM, New York, 2012.
- [21] Y. Lin and S.-T. Yau, Ricci curvature and eigenvalue estimate on locally finite graphs, Math. Res. Lett. 17 (2010), no. 2, 343-356.
- [22] S. Liu, Multi-way dual Cheeger constants and spectral bounds of graphs, arXiv:1401.3147, January 2014.
- [23] S. Liu, An optimal dimension-free upper bound for eigenvalue ratios, arXiv: 1405.2213, May 2014.
- [24] L. Miclo, On eigenfunctions of Markov processes on trees, Probab. Theory Related Fields 142 (2008), no. 3-4, 561-594.
- [25] M. Mimura, Multi-way expanders and imprimitive group actions on graphs, arXiv:1403.2322, March 2014.
- [26] B. Mohar, Eigenvalues, diameter, and mean distance in graphs, Graphs Combin. 7 (1991), no. 1, 53-64.
- [27] M. Tanaka, Multi-way expansion constants and partitions of a graph, arXiv:1112.3434, June 2013.

DEPARTMENT OF MATHEMATICAL SCIENCES, DURHAM UNIVERSITY, DH1 3LE DURHAM,
UNITED KINGDOM

E-mail address: `shiping.liu@durham.ac.uk`

DEPARTMENT OF MATHEMATICAL SCIENCES, DURHAM UNIVERSITY, DH1 3LE DURHAM,
UNITED KINGDOM

E-mail address: `norbert.peyerimhoff@durham.ac.uk`